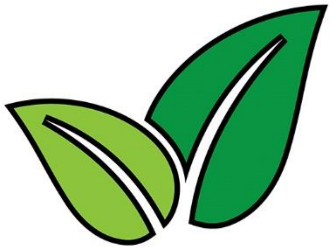
Leafy Math



Spring arrives each year and brings an awakening of plants that have lain dormant throughout the long, gray fall and winter months. The earth moves around our star, and because of the 23.5-degree tilt in the earth's axis, the northern hemisphere leans toward the sun and receives its rays more directly. The winter sun sits low in the sky and its light is smeared out across the land, but the higher sun angles of summer result in a greater intensity of the light—more photons per square inch—and plants respond by sending out shoots and leaves to harvest the light.

This harvest of the sun's light is crucial. Plants are fueled largely by a simple sugar called glucose, and that substance is the result of photosynthesis, one of the most fundamental biological processes on our planet. A simplified representation of photosynthesis looks like this: *sunlight + carbon dioxide + water* ⇨ *glucose + oxygen.*

Now here is an amazing fact: Every year terrestrial and marine plants make enough glucose to fill a freight train 30 million miles long (Hoagland 1995). That's enough to circle the earth more than 1,000 times! Every year! The process requires sunlight, and plants have evolved an array of fascinating strategies and structures that enable them to gather this vital resource. Mathematics gives us a lens that allows us to appreciate some of the wonders of plants as they scoop up sunlight.

It is spring, the middle of May, to be exact, as I write this, and a walk around my neighborhood in Seattle reveals a host of competing and coexisting schemes and structures that plants use to escape the shade and present their leaves to the sun. Closest to the earth are groundcover plants, and that name is quite descriptive. These plants, such as grass and dandelions, spread out and take over the low levels. Rising above the groundcover plants are shrubs and bushes. They grow higher and tend to shade out the lower-level plants. Trees are more extravagant in their use of the same tactic, reaching dozens of meters into the air, intercepting sunlight at levels where only clouds—and other trees—can block them from the sun.

Look closely and you will see that plants, from the ground-covering dandelion to the alder swaying in the breeze, have evolved specialized structures at the macro level to enhance photosynthesis. And here is where mathematics becomes clearly useful in understanding and appreciating the beauty of these structures.

In upper elementary grades and in middle school, we work with symmetry, concentrating on line-symmetry and rotational symmetry. When one-half of a structure is the reflection of the other half across a central line, we have line or reflection symmetry. When an element of a structure is repeated one or more times as we rotate around an axis, we have rotational symmetry.

Take a look at the four plants shown in figure 1. The symmetry, although sometimes a little bit approximate, is hard to miss. Now that I am aware of it, I see symmetry everywhere—especially in the leaves but often in the arrangement of leaves on twigs and branches in nearly every plant I see. And the ways in which these patterns in growth help plants to present leaves to the sun are remarkable. For example, the butterfly bush shown in figure 1d sends out two leaves directly opposite one another and then farther along the stem, another pair, also opposite one another; but the second set is at 90 degrees from the lower set. This way the leaves don't shade each other out. This is a *decussate* leaf pattern (Burakoff 2019). Can you imagine what a *tricussate* pattern might be? What angles might be involved?

Fig. 1 Different patterns of growth optimize the ways that plants present leaves to the sun.

(a) Oregon Grape

(b) Rhododendron

(c) Corn Lilly

(d) Butterfly Bush

The Rhododendron in figure 1b has leaves that radiate out from the central stem, but space between the leaves allows light to make its way to those further down, and this serves the plant well: It can grow to be quite tall in some species. This arrangement of leaves allows the whole plant to gather the sunlight that enables photosynthesis. What sorts of symmetry do you see in the other plants shown in figure 1?

In the remainder of this article, I will concentrate on one plant, the western bracken fern (*Pteridium aquilinum)*, and my interpretation of the fascinating strategy it employs in its quest for sunlight, involving both symmetry and structure. Along the way, we will find good uses for mathematics in developing a more complete understanding of how this happens.

Fig. 2 Bracken Fern Side View

Fig. 3 Bracken Fern Top View

In figure 2 you see an example of a bracken fern from a side view. This one was found near Shelton, Washington, but they are not hard to find; they grow in open areas, often along roadsides, in nearly every state in the union and across a wide range of climate zones throughout North America and around the world. Figure 3 shows a bracken fern from above. Even from a distance, the symmetry in this plant is striking. Now take a close look at the two photographs and consider the way in which the structure of the plant enables it to gather sunlight. What do you notice about the way in which the plant rises from the ground as seen in the side view presented in figure 2? What about the way the "leaves" sometimes overlap and sometimes don't when viewed from the top as in figure 3? How might you begin to apply some measurement and mathematics to really understand these things more fully? I encourage you to take a few minutes to think about these things before you read on.

Before I continue, let me say that although the parts of the fern have proper scientific names (stalk, petiole, frond, pinnule, and so forth), I am not going to use them because that is not the point of this analysis. I consider it good to call things by their proper names, but I want to get straight to the math. More importantly in a classroom, I would not want to insist on the use of scientific nomenclature if I thought it would discourage students from diving in with the application of mathematics. Proper names should be introduced when they are needed. With that said, let's take a mathematical look at the fern.

The "trunk" of the fern makes approximately a 90-degree angle with the ground (assuming the ground is flat). This makes sense as it affords the fern the best chance to rise above the other plants and get to the sunlight. At this angle, every inch of growth puts the fern an inch above other plants that might shade it out. The first branches come from the trunk while it is still in this vertical mode, but soon thereafter the trunk begins to bend; and by the third or fourth branches, the plant is often bent over 90 degrees or more, running parallel to the ground. This is especially true once spring is over and the plant has reached a mature stage. The leaves, at first quite large, get smaller and smaller, and the trunk becomes thinner as the whole plant narrows to its triangular finish. If you look carefully at figure 2, you will see that the first set of branches and their leaves are overlapped by those above them. You can see it even more clearly in figure 3; moving from the bottom of the figure toward the top, the lower set of leaves is overlapped by the one above and beyond it as you move along the trunk, and that set of leaves is overlapped by the next one, but to a lesser extent. In fact, looking farther along the trunk, the branches are found closer to one another, but the leaves overlap less and less as you proceed to the end.

This constitutes a sort of pattern. I have looked at many bracken ferns (too many, I've been told, but that's another story), and I can tell you that this is a consistent pattern. Your students could take a look at some ferns if some are available (call it homework) or do an image search if some are not but the internet is.

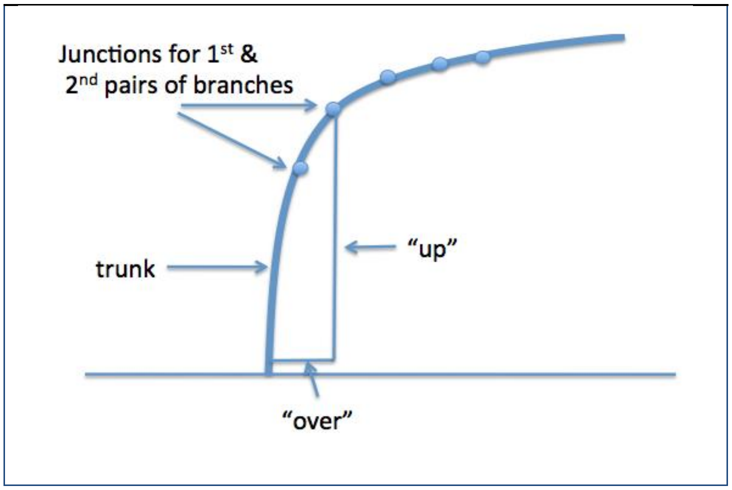
Can you think of a way to use mathematics to make quantitative sense of this change in the degree of overlapping as you move along the trunk? The Common Core State

Standards for mathematics direct us (as the NCTM standards did before them) to enable our students to pay attention to patterns and to develop their ability and inclination to use math to make sense of the world. This is a word problem that the world has written for us!

Here is what I did: I visited 10 bracken ferns and spent about 15 minutes with each one, measuring first the height at which each of the first seven branches emerged from the trunk. I made a column on a sheet of notebook paper and labeled it *up*. The *up* number is simply the height, the elevation at which the branch first leaves the trunk, measured up from a spot on the ground directly beneath the junction on the trunk.

Then I measured the distance from this point on the ground back to the place where the trunk leaves the ground. This became my *over* number. Up and over are shown in figure 4 for the second set of branches. Taken together, these two numbers tell me how far up and how far over each junction is from the base of the fern. They are essentially the coordinates of the junction in a two-dimensional representation of the plant, where the origin is the point at which the trunk emerges from the ground.

Fig. 4 The "up" and "over" numbers for the second set of branches tell us how far each junction is from the base of the fern.



Next I measured the distance between the branches progressing from the first set to the second, from the second to the third, and so on moving out along the trunk as far as the seventh set of branches. Finally, I measured the length of the leaves leaving the branches. I did this for the first full-sized leaves emitted from each of the first seven sets of branches (see figure 5).

I should be clear that this is not all I did. I started out looking at a variety of other factors, such as the angle at which the branches emerged from the trunk, the lengths of the branches, the complexity of the leaves and leaflets, and so forth. This "messing about" led me to understand what I wanted to look at. I noticed in this process that there was no end of other factors that might have been fascinating to explore but that they were more or less interesting to me according to my grasp of the mathematics needed to pursue the investigation. This is true for our students as well and points to the importance of differentiation in our curriculum.

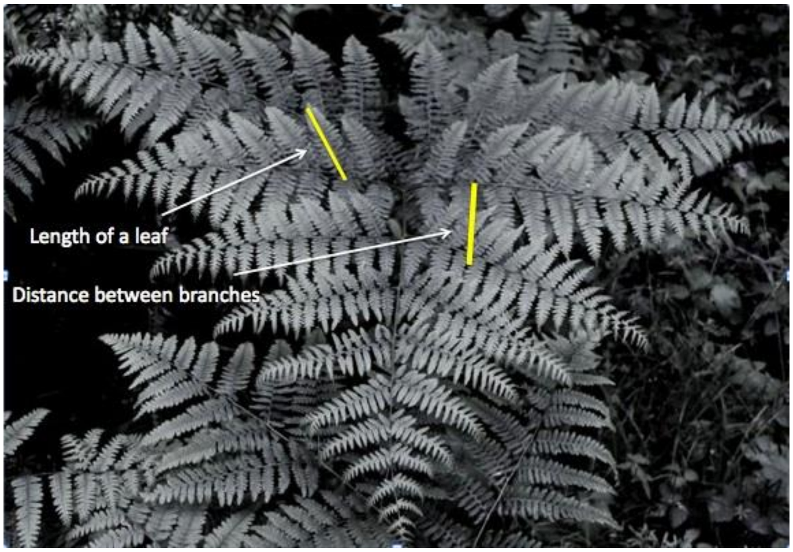


Fig. 5 Leaf Lengths and Distances between Branches

If this has made sense, you can see that now I am in a position to calculate what I called the *overlap factor*. This is the degree to which the leaves coming from succeeding branches overlapped one another and therefore had the potential to cast shade on one another, inhibiting photosynthesis. When the sum of the lengths of the leaves from two successive branches is greater than the distance between the branches, the ratio of this sum and the distance is greater than one. An overlap factor of two, for example results when the leaves are twice as long, taken together, as the gap between successive branches. An overlap factor less than one means that the leaves are not long enough to reach each other, to fill the space between successive branches. You will also see that with my up and over figures, I can calculate an approximation to the slope of the trunk at the points where the first through sixth junctions occurred.

When all of the numbers have been crunched and calculated, we end up with the results seen in Table 1. I have to say that it was a lot more fun to do the measurements and the calculations than it is to look at the finished product, boiled down to these few figures. It is a little like a cross-country race: You get a lot more from running one than you do from standing at the finish line and watching the runners come in.

Table 1

Branch	1st	2nd	3rd	4th	5th	6th
Mean Slope (“up” ÷ “over”)	5.6	1.5	0.8	0.5	0.4	0.2

	1st–2nd	2nd–3rd	3rd–4th	4th–5th	5th–6th	6th–7th
Mean Overlap Factor	1.7	1.5	1.4	1.2	1.1	0.9

Nevertheless, we can see from the numbers that as we go up and then out along the curving trunk, from the first branches to the last, the slope, at first great, becomes more and more shallow and the overlap of the leaves becomes less and less pronounced until finally they do not, on average, meet one another. This makes sense because when the slope is large, the trunk is still shooting upward and the leaves on successive sets of branches are far enough above one another that sunlight will get through. Whereas by the time the slope is nearing zero, the overlap factor is near one—no overlap. This is good for the fern because any overlap would result in one leaf laying right atop and shading out the other. In the race to secure the sun's energy and produce the sugars needed to build and grow, the bracken fern has developed a strategy that seems to work.

What we as math teachers need are teaching strategies that enable our students to harvest the energy that mathematics can bring to their lives. I do believe that being able to use math to make sense of the world is the long-term goal of teaching the subject and that when we get there, our students will understand why math is worth knowing.

Lesson Plan
Learn more about implementing Leafy Math in your classroom by exploring the Illuminations lesson here! Then, share your experiences using Math Sightings on social media with the hashtag #MathSightings.

References

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Resources

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